UCI University of California, Irvine

CS 277: Control and Reinforcement Learning

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Lecture 3: Temporal-Difference Methods

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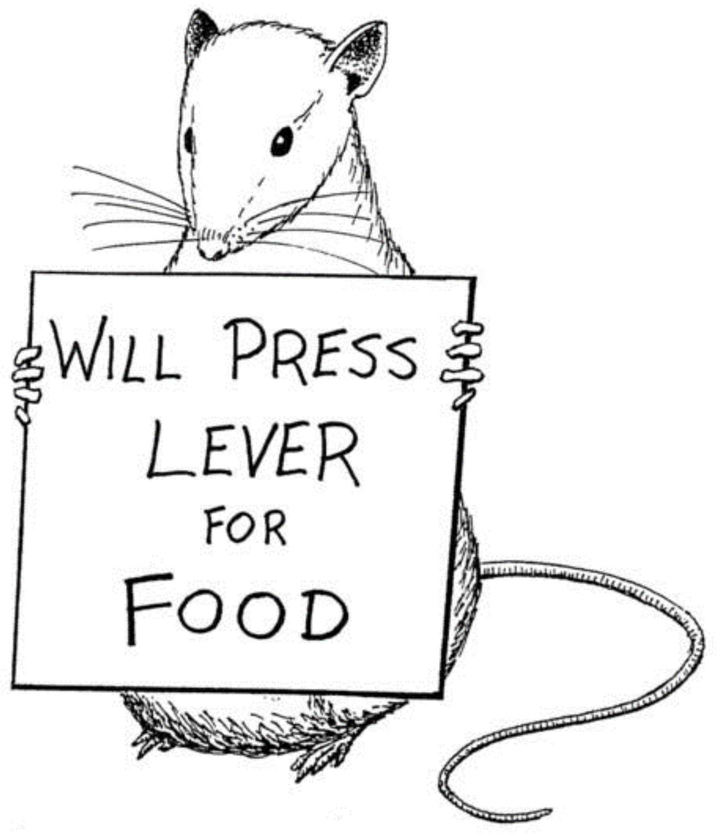


Logistics

assignments

- Exercise 1 due Friday
- Quiz 2 to be published soon, due next Monday

resources

- Lots of resources on the website
- Will be updated with papers relevant to each lecture

Today's lecture

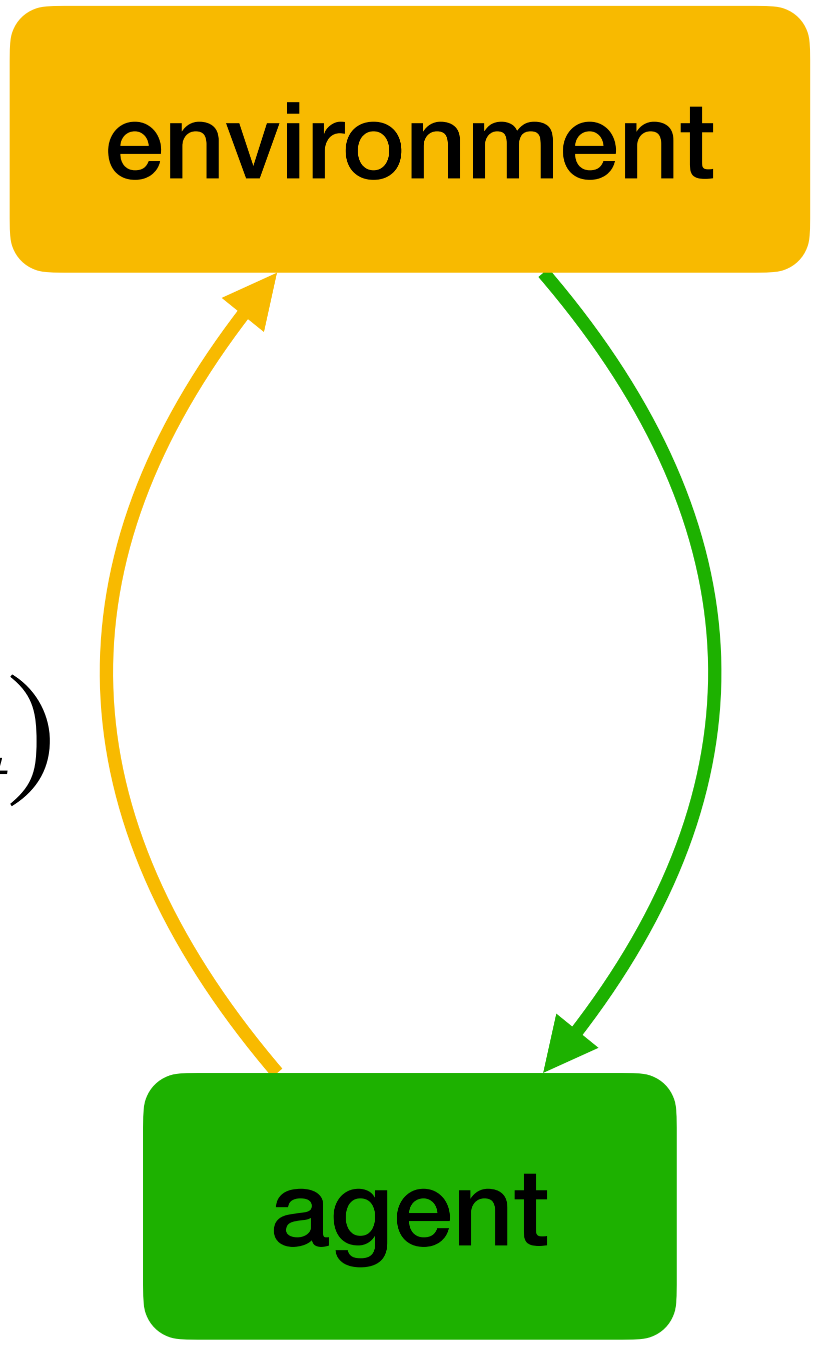
Policy evaluation

Temporal Difference

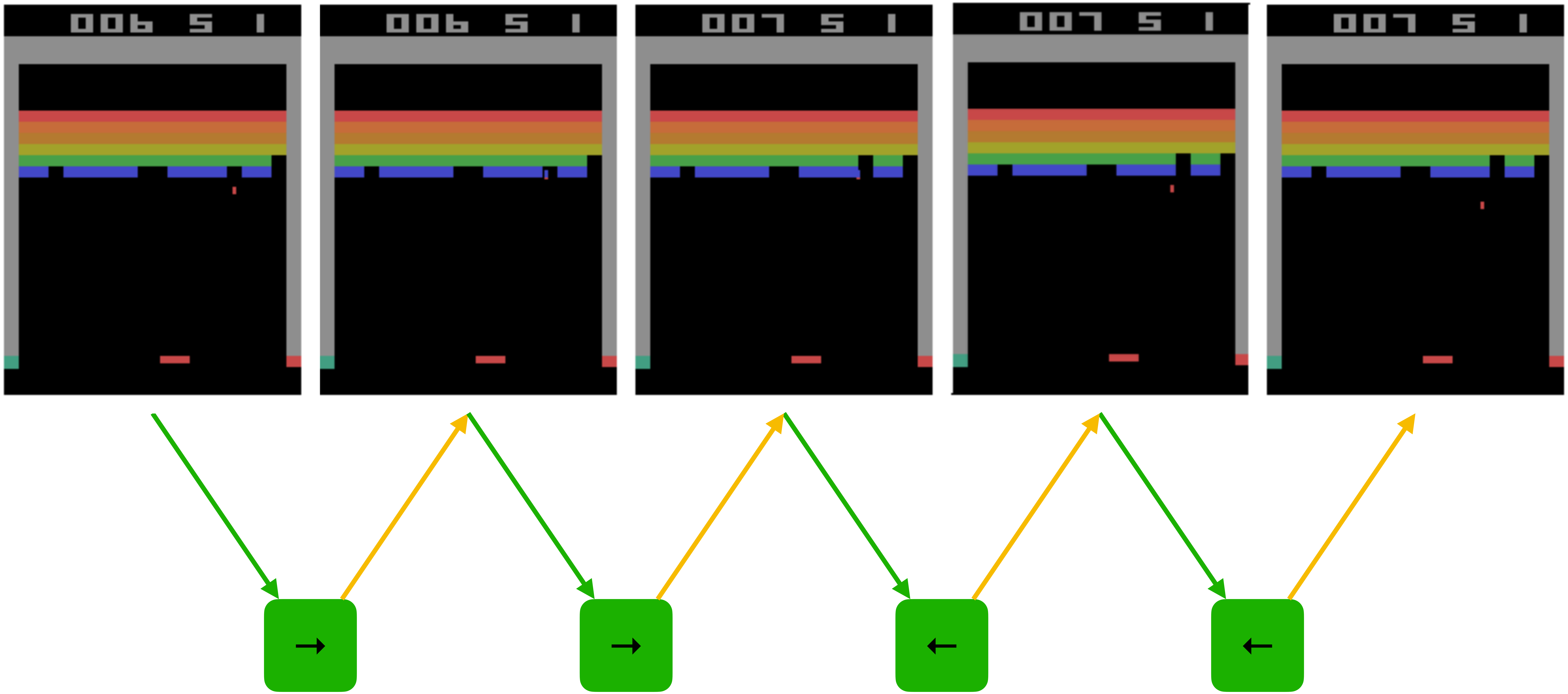
Policy improvement

Basic RL concepts

- State: $s \in S$; action: $a \in A$; reward: $r(s, a) \in \mathbb{R}$
- Dynamics: $p(s_{t+1} | s_t, a_t)$ for stochastic; $s_{t+1} = f(s_t, a_t)$ for deterministic
- MDP: $\mathcal{M} = \langle S, A, p \rangle$ or $\langle S, A, p, r \rangle$
- Policy: $\pi(a_t | s_t)$ for stochastic; $a_t = \pi(s_t)$ for deterministic
- Trajectory: $p_\pi(\xi = s_0, a_0, s_1, a_1, \dots) = p(s_0) \prod_{t \geq 0} \pi(a_t | s_t) p(s_{t+1} | s_t, a_t)$
- Return: $R(\xi) = \sum_{t \geq 0} \gamma^t r(s_t, a_t) \qquad 0 \leq \gamma < 1$



Example: Breakout



reward: **+0** **+1** **+0** **+0**

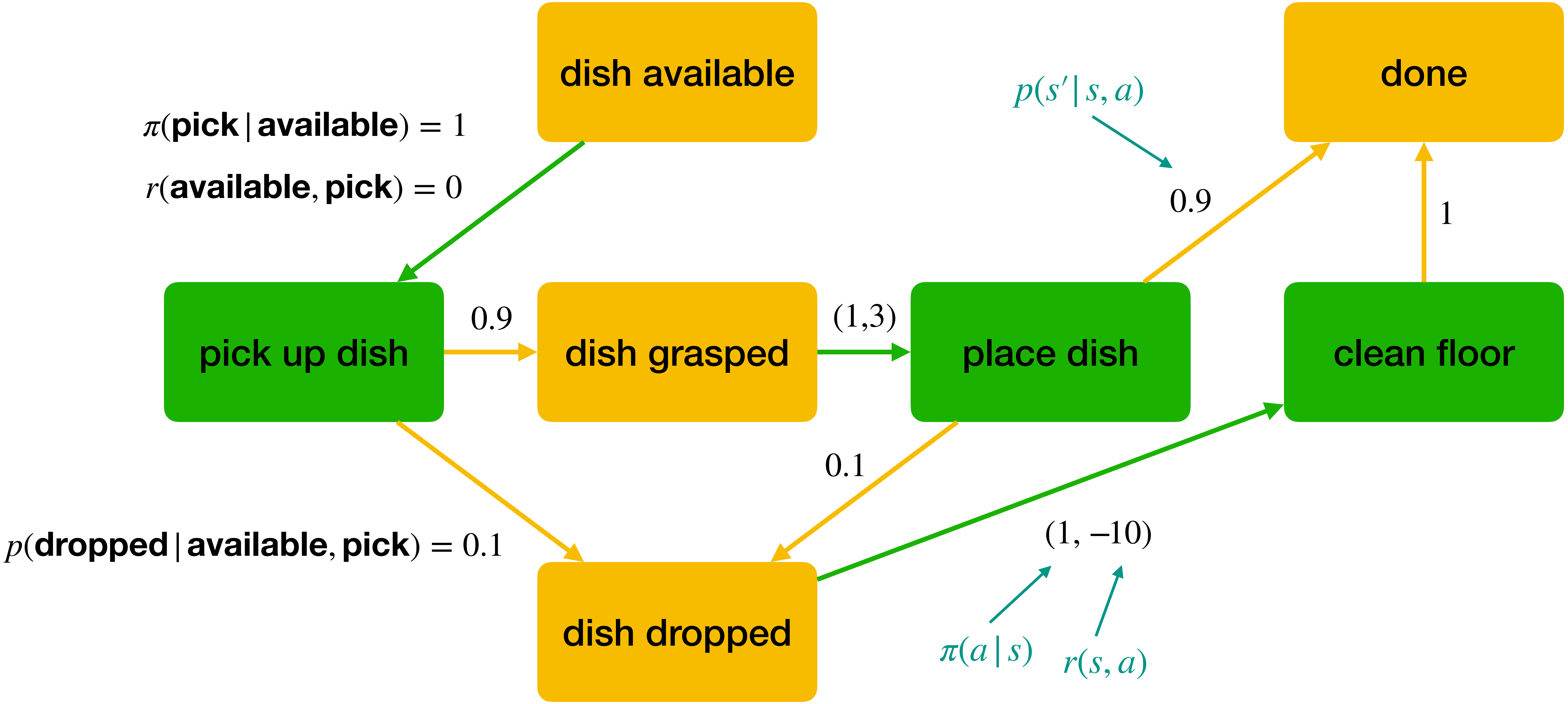
Formulating reward: considerations

- We define $r(s,a)$, is that general enough?
- What if the reward depends on the next state s'?
  - If we only care about expected reward, define $r(s,a) = \mathbb{E}_{(s'|s,a)\sim p}[r(s,a,s')]$
- What if the reward is a random variable $\tilde{r}$?
  - Define $r(s,a) = \mathbb{E}[\tilde{r} \,|\, s,a]$
  - In practice we see $\tilde{r}$ ⇒ don't just assume you know $r(s,a) = \tilde{r}$

RL objective: expected return

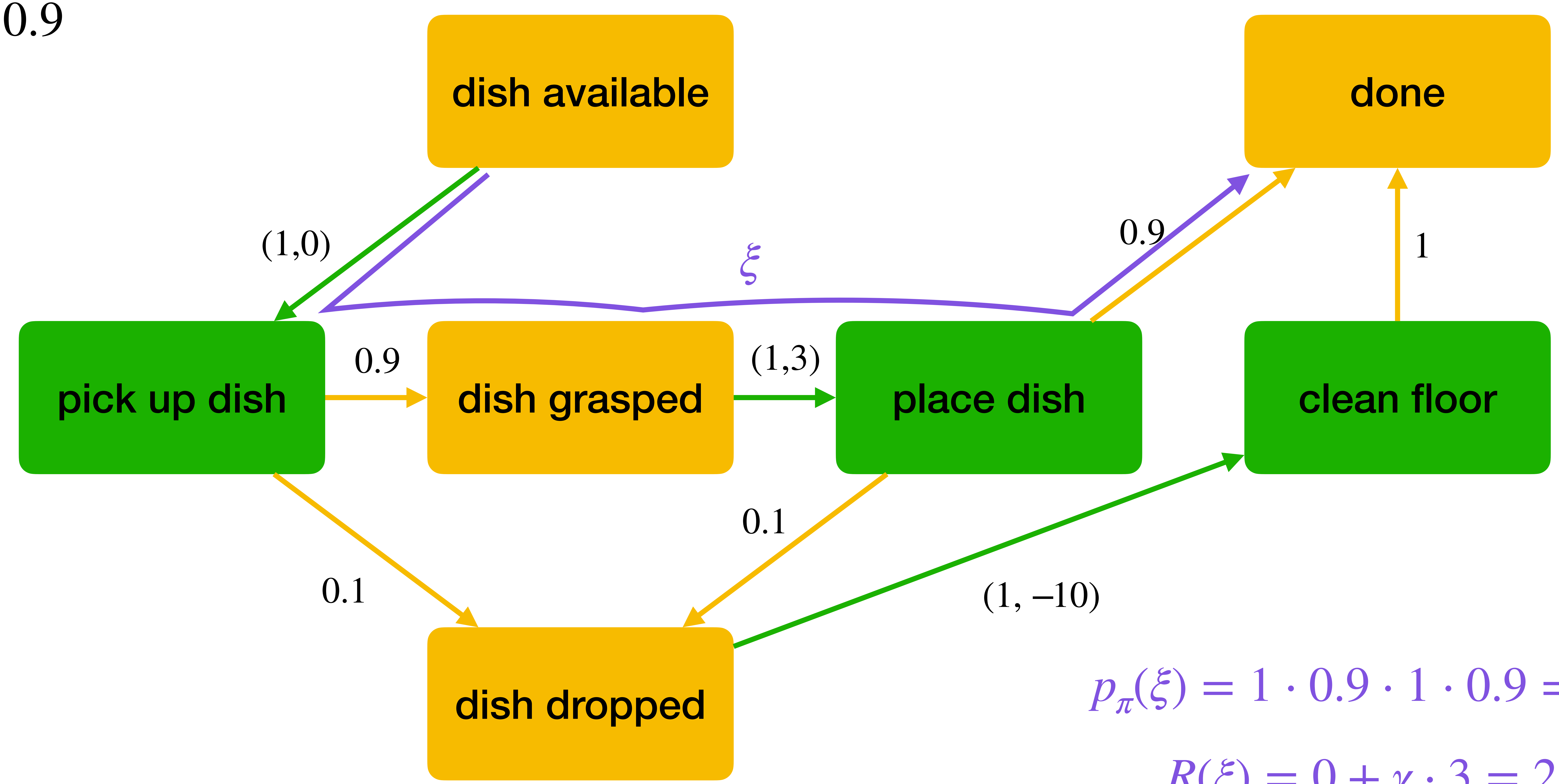
- We need a scalar to optimize
- Step 1: we have a whole sequence of rewards $\{r_t = r(s_t, a_t)\}_{t \geq 0}$
    - Summarize as return $R(\xi) = \sum_{t \geq 0} \gamma^t r(s_t, a_t)$
- Step 2: $R(\xi)$ is a random variable, induced by $p_\pi(\xi)$
    - Take expectation $J_\pi = \mathbb{E}_{\xi \sim p_\pi}[R(\xi)]$
- J_π can be calculated and optimized

Policy evaluation: example



Policy evaluation: example

$\gamma = 0.9$

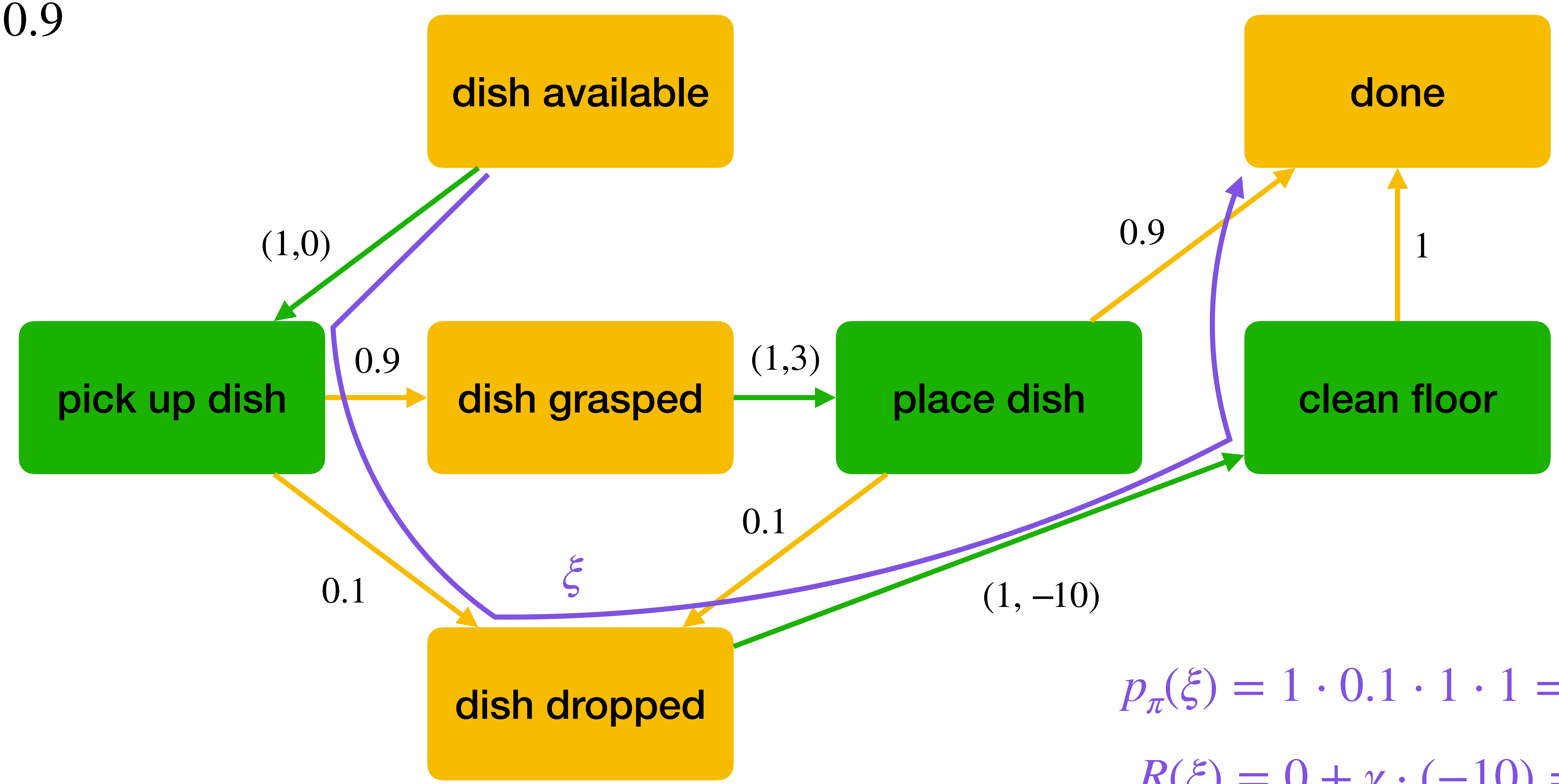


$p_\pi(\xi) = 1 \cdot 0.9 \cdot 1 \cdot 0.9 = 0.81$

$R(\xi) = 0 + \gamma \cdot 3 = 2.7$

Policy evaluation: example

$\gamma = 0.9$



$p_\pi(\xi) = 1 \cdot 0.1 \cdot 1 \cdot 1 = 0.1$

$R(\xi) = 0 + \gamma \cdot (-10) = -9$

Monte Carlo (MC) policy evaluation

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- Computing $J_\pi = \mathbb{E}_{\xi \sim p_\pi}[R(\xi)] = \sum_\xi p_\pi(\xi) R(\xi)$ can be hard
  - Exponentially many trajectories
  - Model-based = requires $p(s' | s, a)$, which may not be known
- Monte Carlo: estimate expectation using empirical mean

$$J_\pi \approx \frac{1}{m} \sum_i R(\xi^{(i)}) \qquad \xi^{(i)} \sim p_\pi$$

  - Model-free = can sample with rollouts, without knowing p

MC: iterative computation

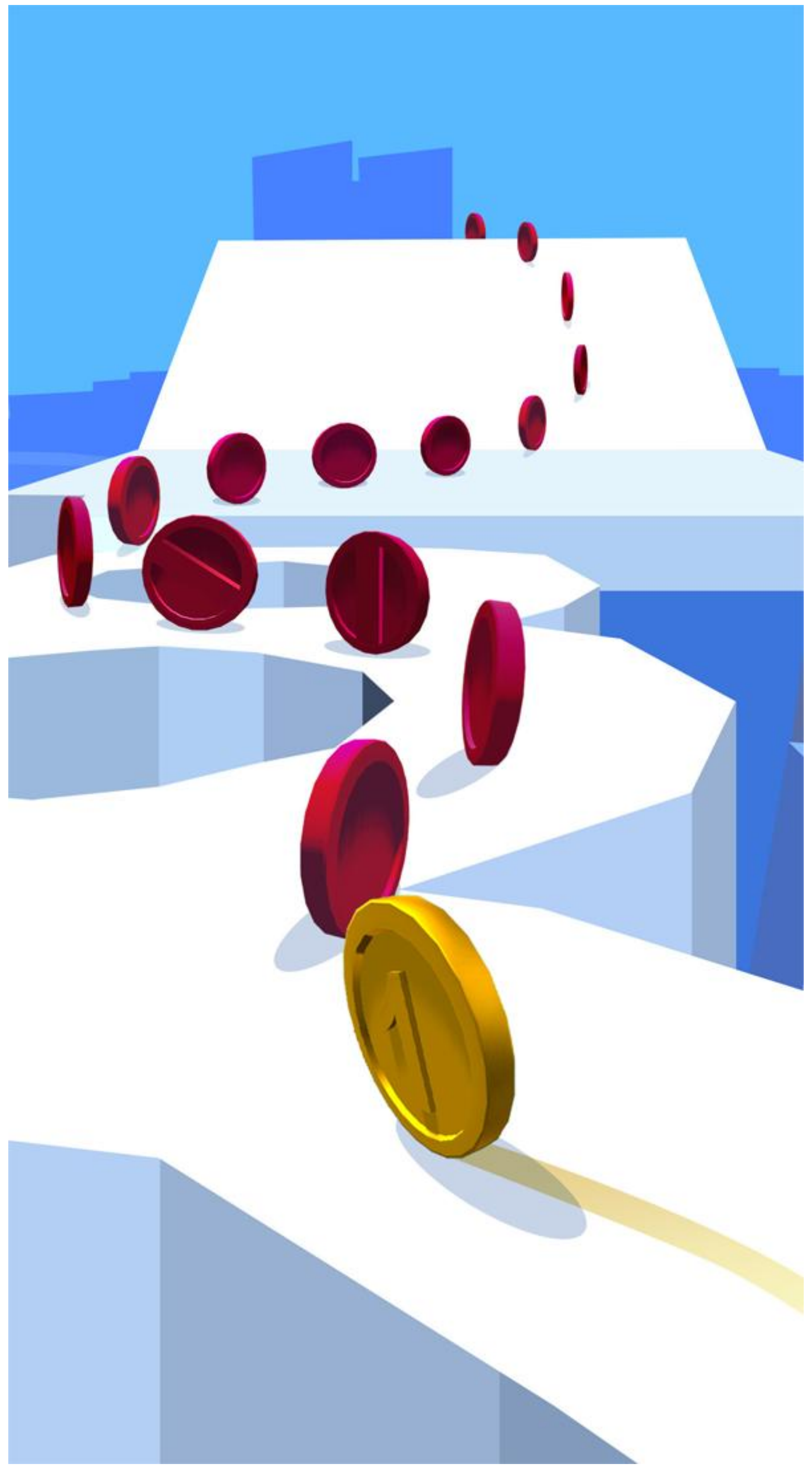
MF

- We can keep a running average $\bar{R}^{(i)}$ of the first i returns
  - Update: $\bar{R}^{(i)} = ((i-1)\bar{R}^{(i-1)} + R(\xi^{(i)}))\frac{1}{i}$
  - More generally: $\bar{R}^{(i)} = (1-\alpha)\bar{R}^{(i-1)} + \alpha R(\xi^{(i)}) = \bar{R}^{(i-1)} + \alpha(R(\xi^{(i)}) - \bar{R}^{(i-1)})$ — residual: $R(\xi^{(i)}) - \bar{R}^{(i-1)}$
  - α is a learning rate, exact average when it vanishes harmonically as $\frac{1}{i}$
- To simplify expressions, we denote this update: $J \to_\alpha R(\xi)$
  - Read: update J toward $R(\xi)$ at rate α

Value function

- RL objective: maximize expected return $J_\pi = \mathbb{E}_{\xi \sim p_\pi}[R]$
- We don't control s_0, can break down: $J_\pi = \mathbb{E}_{s_0 \sim p}[V_\pi(s_0) \,|\, s_0]$
  - with the value function $V_\pi(s) = \mathbb{E}_{\xi \sim p_\pi}[R \,|\, s_0 = s]$
- $V_\pi(s)$ is the expected reward-to-go (= future return):
  - For any t_0, define $R_{\geq t_0} = \sum_{t \geq t_0} \gamma^{t-t_0} r(s_t, a_t)$
  - Then $V_\pi(s) = \mathbb{E}_{\xi \sim p_\pi}[R_{\geq t_0} \,|\, s_{t_0} = s]$

future reward after being in state s in time t_0



MC for value-function estimation

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Algorithm MC for value-function estimation

Initialize $V(s) \leftarrow 0$ for all $s \in S$
repeat
  Sample $\xi \sim p_\pi$
  Update $V(s_0) \to R(\xi)$

- Why not use the same samples for non-initial states?

Algorithm MC for value-function estimation (version 2)

Initialize $V(s) \leftarrow 0$ for all $s \in S$
repeat
  Sample $\xi \sim p_\pi$
  Update $V(s_t) \to R_{\geq t}(\xi)$ for all $t \geq 0$

MC with function approximation

- What if the state space is large?
  - Can't represent $V(s)$ as a big table
  - Won't have enough data to estimate each $V(s)$
- Function approximation: represent $V_\theta : S \to \mathbb{R}$
  - $\theta \in \Theta$, a parametric family of functions; for example, a neural network
- Generalization over state space $\Rightarrow$ data efficiency

Algorithm MC with function approximation

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Initialize V_θ
repeat
  Sample $\xi \sim p_\pi$
  Descend on $\mathcal{L}_\theta = \sum_{t\geq 0}(R_{\geq t}(\xi) - V_\theta(s_t))^2$

with tabular representation:

$$V(s_t) \mathrel{+}= -\alpha \nabla_{V(s_t)} \mathcal{L} = 2\alpha(R_{\geq t}(\xi) - V(s_t))$$

same as in previous slide

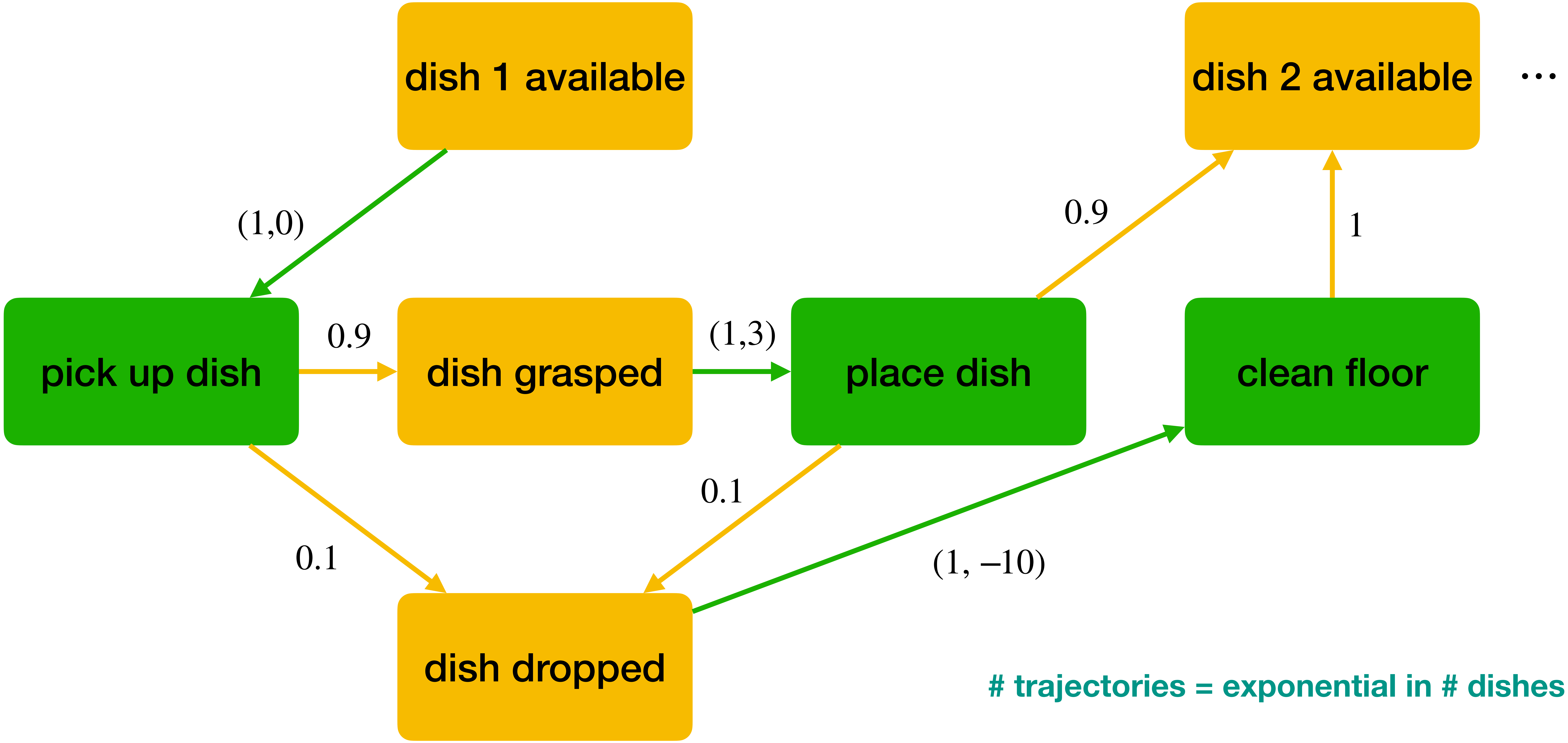
Today's lecture

Policy evaluation

Temporal Difference

Policy improvement

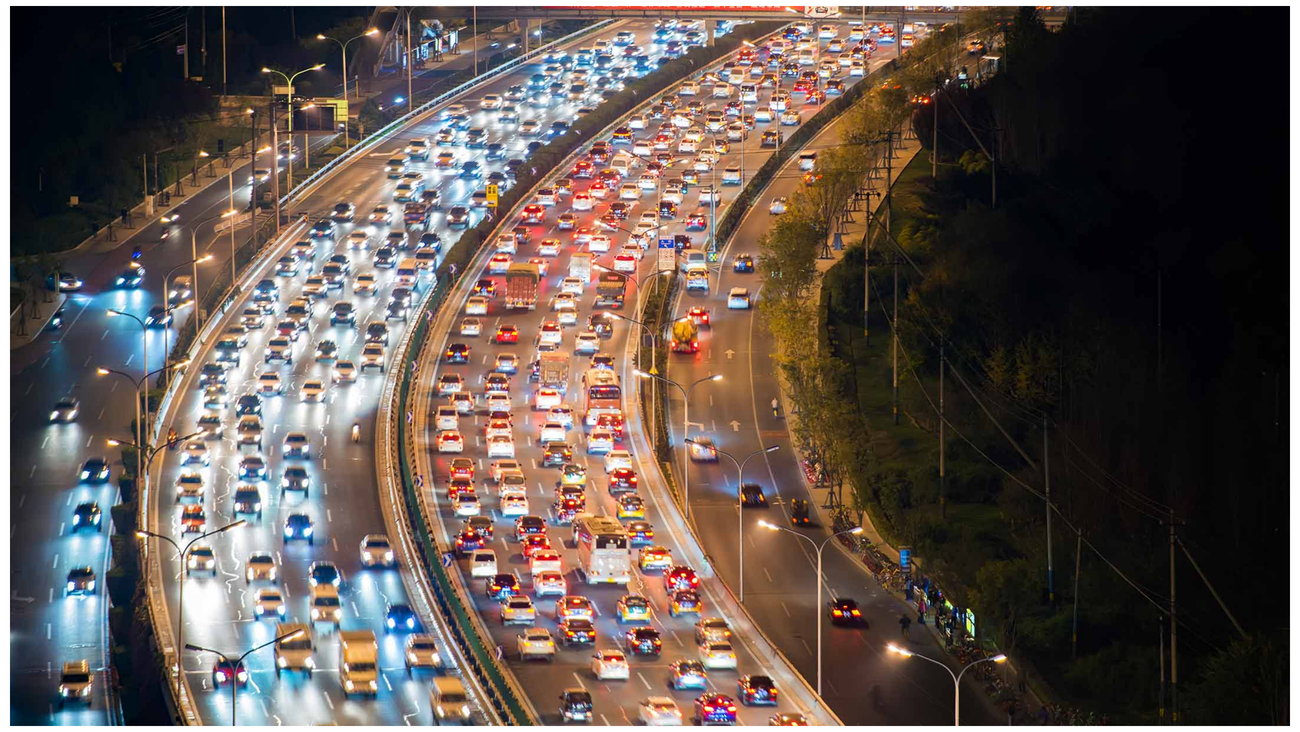
Policy evaluation: example



trajectories = exponential in # dishes

MC inefficiency

- The MC estimator is unbiased (correct expectation), but high variance
  - Requires many samples to give good estimate
- But MC misses out on the sequential structure



- Example:
  - Day 1: I take route 1 to work — 40 minutes; I take route 2 home — 10 minutes
  - Day 2: I take route 3 to work — 30 minutes; I take route 4 home — 30 minutes
- Which route should I take to work?
  - Route 1 → 50-minute daily commute, route 3 → 60-minute; is route 1 better?

Dynamic Programming (DP)


Richard Bellman

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- Dynamic Programming = remember reusable partial results
- Value recursion:

$$V_\pi(s) = \mathbb{E}_{\xi\sim p_\pi}[R \,|\, s_0 = s]$$

break down sum of rewards

$$= \mathbb{E}_{\xi\sim p_\pi}[r(s_0, a_0) + \gamma R_{\geq 1} \,|\, s_0 = s]$$

first reward only depends on a

$$= \mathbb{E}_{(a|s)\sim\pi}[r(s, a) + \gamma \mathbb{E}_{\xi\sim p_\pi}[R_{\geq 1} \,|\, s_0 = s, a_0 = a]]$$

s' is a state, all that matters for $R_{\geq 1}$

$$= \mathbb{E}_{(a|s)\sim\pi}[r(s, a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[\mathbb{E}_{\xi\sim p_\pi}[R_{\geq 1} \,|\, s_1 = s']]]$$

definition of $V_\pi(s')$

$$= \mathbb{E}_{(a|s)\sim\pi}[r(s, a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[V_\pi(s')]]$$

[Bellman, 1956]

Policy evaluation: example

$$V_\pi(s) = \mathbb{E}_{(a|s)\sim\pi}[r(s,a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[V_\pi(s')]]$$

$\gamma = 0.9$

$V(\textbf{done}) = 0$

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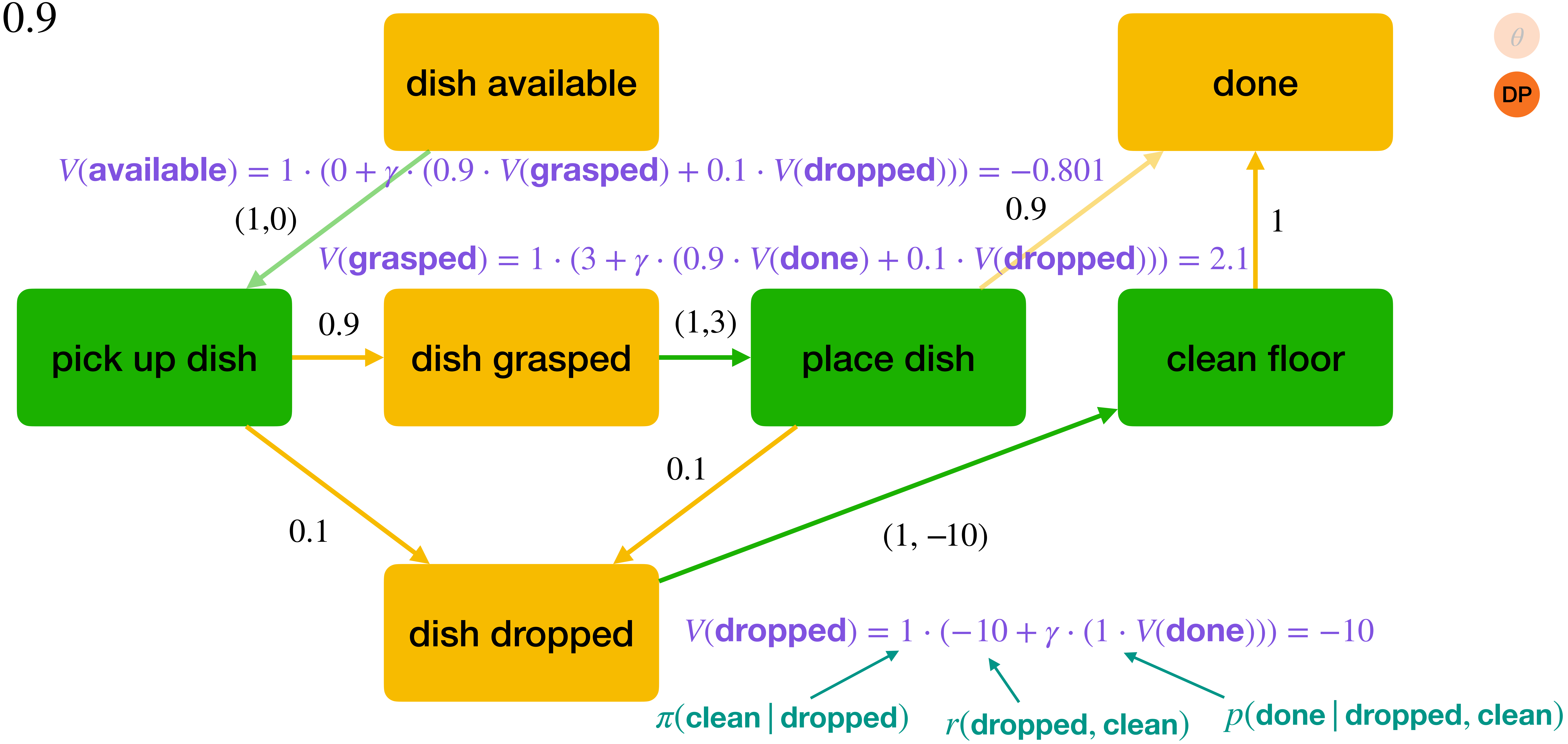

DP + MC: Temporal Difference (TD)

- Policy evaluation with DP: $V_\pi(s) = \mathbb{E}_{(a|s)\sim\pi}[r(s,a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[V_\pi(s')]]$
  - Drawback: model-based = need to know p

recursion from s' to s
= backward in time!

- MC: $V(s) \to R_{\geq t}(\xi)$, where $\xi \sim p_\pi$ and $s_t = s$
  - Drawback: high variance

- Put together: $V(s) \to r + \gamma V(s')$
  - where $s = s_t$, $r = r(s_t, a_t)$, and $s' = s_{t+1}$ in some trajectory
  - In other words: $V(s) \leftarrow V(s) + \alpha(r + \gamma V(s') - V(s))$

temporal difference
between $V(s')$ and $V(s)$

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Q function

- To approach V_π when we update $V(s) \to r + \gamma V(s')$, we need on-policy data
  - Roll out π to see transition $(s, a) \to s'$ with reward r
- On-policy data is expensive: need more every time π changes
- Action-value function: $Q_\pi(s, a) = \mathbb{E}_{\xi \sim p_\pi}[R \,|\, s_0 = s, a_0 = a]$
  - Compare: $V_\pi(s) = \mathbb{E}_{\xi \sim p_\pi}[R \,|\, s_0 = s] = \mathbb{E}_{(a|s) \sim \pi}[Q_\pi(s, a)]$

- Action-value backward recursion: $Q_\pi(s, a) = r(s, a) + \gamma \mathbb{E}_{(s'|s,a) \sim p}[V_\pi(s')]$
  - Broke down $V_\pi(s) = \mathbb{E}_{(a|s) \sim \pi}[r(s, a) + \gamma \mathbb{E}_{(s'|s,a) \sim p}[V_\pi(s')]]$ into two parts

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TD from off-policy data

- Backward recursion in two parts:

$$V_\pi(s) = \mathbb{E}_{(a|s)\sim\pi}[Q_\pi(s,a)] \qquad Q_\pi(s,a) = r(s,a) + \gamma\mathbb{E}_{(s'|s,a)\sim p}[V_\pi(s')]$$

- This should hold in every state and action
  - (s,a) can be sampled from any distribution $p_{\pi'}$ for any alternative π'
- Put together, we update $Q(s,a) \to r + \gamma\mathbb{E}_{(a'|s')\sim\pi}[Q(s',a')]$
  - For any distribution of (s,a), giving reward r and following state $s' \sim p(\cdot\,|\,s,a)$
  - In other words: $Q(s,a) \leftarrow Q(s,a) + \alpha(r + \gamma\mathbb{E}_{(a'|s')\sim\pi}[Q(s',a')] - Q(s,a))$

temporal difference

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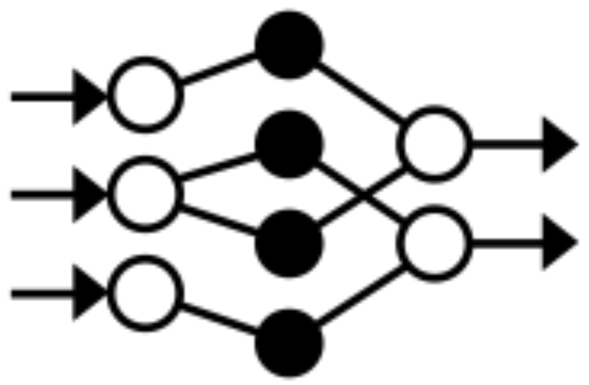
TD with function approximation

- With large state space: represent $V_\theta : S \to \mathbb{R}$ or $Q_\theta : S \times A \to \mathbb{R}$



- Instead of the update $V(s) \to r + \gamma V(s')$
  - Descend on square loss $\mathcal{L}_\theta = (r + \gamma V_{\bar\theta}(s') - V_\theta(s))^2$
  - On on-policy experience (s, a, r, s')

only learn $V_\theta(s)$
$V_{\bar\theta}(s')$ is the target
$\Rightarrow$ don't take its gradient!

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- Instead of the update $Q(s, a) \to r + \gamma \mathbb{E}_{(a'|s') \sim \pi}[Q(s', a')]$
  - Descend on square loss $\mathcal{L}_\theta = (r + \gamma \mathbb{E}_{(a'|s') \sim \pi}[Q_{\bar\theta}(s', a')] - Q_\theta(s, a))^2$
  - On off-policy experience (s, a, r, s')

only learn $Q_\theta(s, a)$
$Q_{\bar\theta}(s', a')$ is the target
$\Rightarrow$ don't take its gradient!

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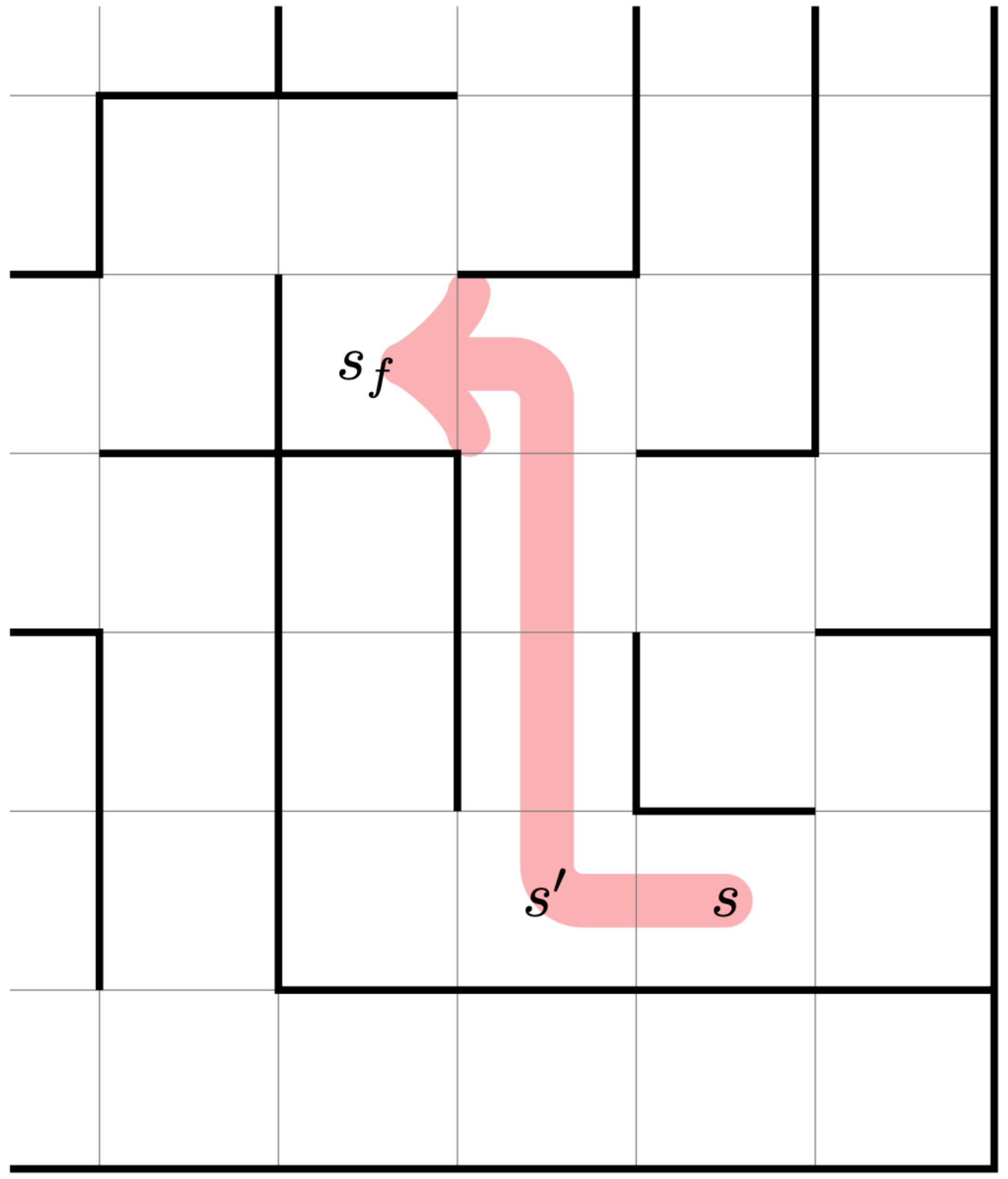
Today's lecture

Policy evaluation

Temporal Difference

Policy improvement

Special case: shortest path



- Deterministic dynamics: in state s, take action a to get to state $s' = f(s, a)$
    - Example above: $s' = f(s, a_{\text{left}})$
- Reward: (-1) in each step (until the goal s_f is reached)

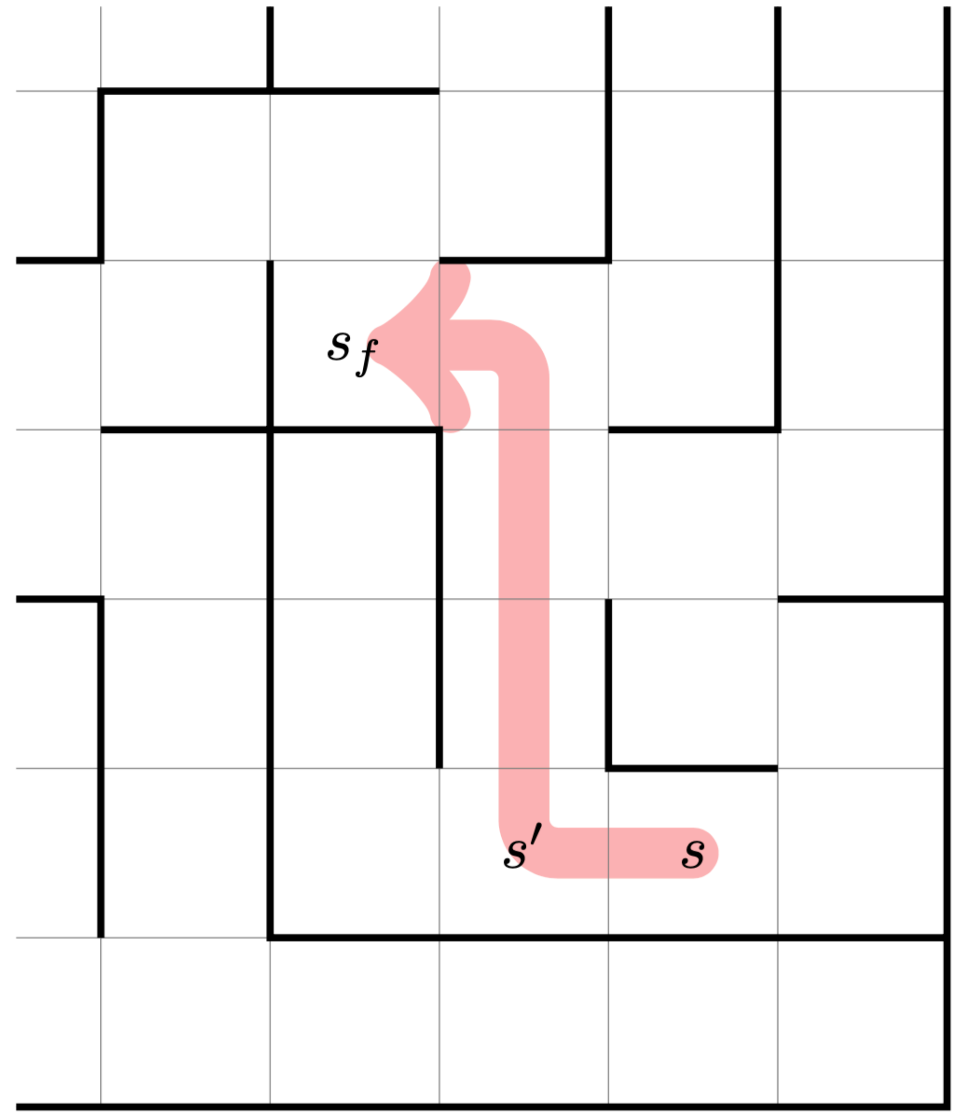
Shortest path: optimality principle

- Proposition: ξ is shortest from s to s_f through s' $\Rightarrow$ suffix of ξ is shortest from s' to s_f

- Proof: otherwise, let ξ' be a shorter path from s' to s_f, then take $s \xrightarrow{\xi} s' \xrightarrow{\xi'} s_f$

- The proposition is "if" but not "only if", because we don't know which s' is best
  - Try them all: for each a, try $s' = f(s, a)$

- Let $V(s)$ be the shortest path length from s to s_f
  - For each candidate s', the shortest path through it is $1 + V(s')$
  - For all $s \neq s_f$, we have $V(s) = \min_a (1 + V(f(s, a)))$

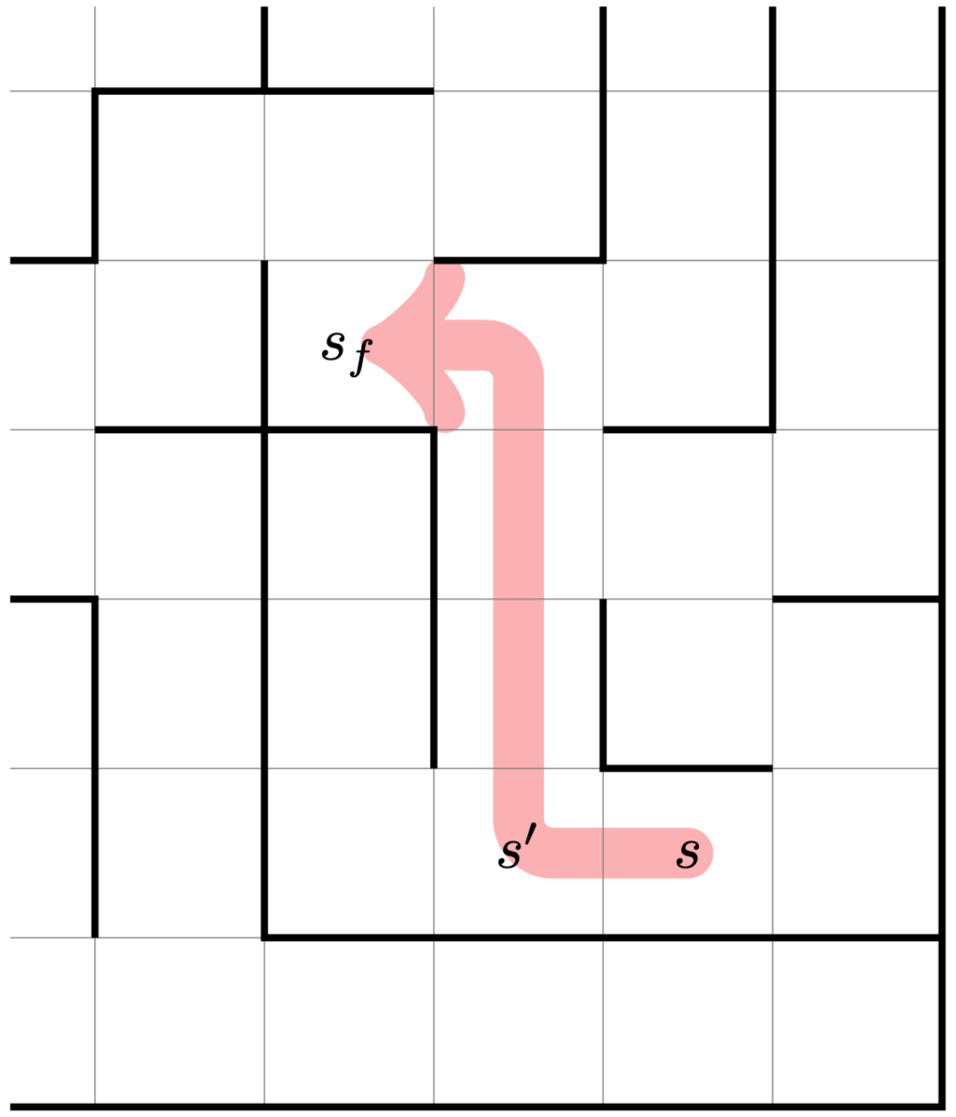


Bellman-Ford shortest path algorithm

- For all $s \neq s_f$, we have $V(s) = \min_a (1 + V(f(s, a)))$

Algorithm Bellman-Ford

$V(s_f) \leftarrow 0$
$V(s) \leftarrow \infty$ for each non-terminal state s
for $|S| - 1$ iterations
  for each non-terminal state s
    $V(s) \leftarrow \min_{a \in A}(1 + V(f(s, a)))$



- The optimal policy is $\pi(s) = \arg\min_a (1 + V(f(s, a)))$

[Ford and Fulkerson, 1962]

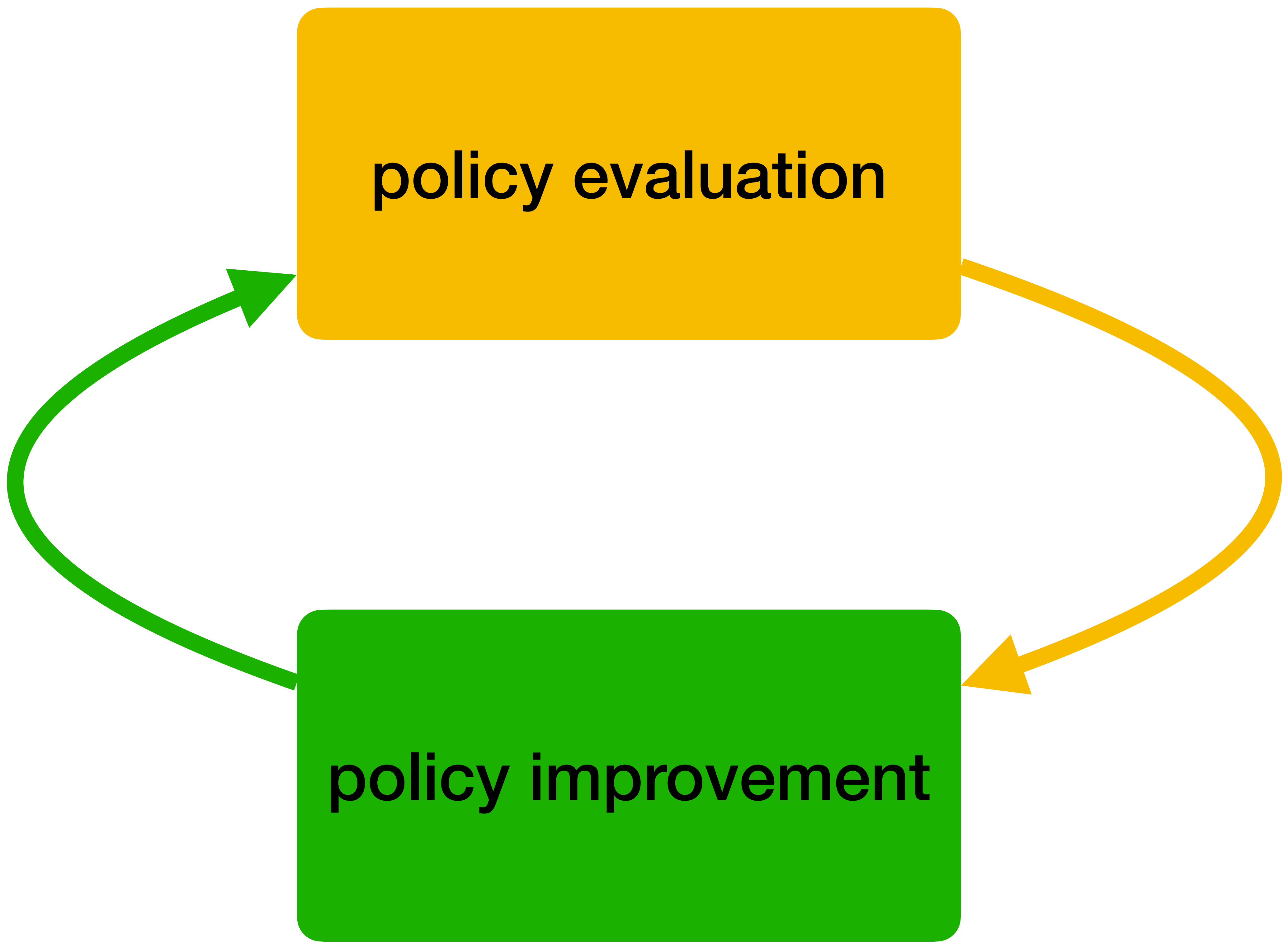
Policy improvement

- A value function suggests the greedy policy:

$$\pi(s) = \arg\max_a Q(s,a) = \arg\max_a (r(s,a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[V(s')])$$

- The greedy policy may not be the optimal policy $\pi^* = \arg\max_\pi J_\pi$
  - But is the greedy policy always an improvement?
- Proposition: the greedy policy for Q_π (value of π) is never worse than π
- Corollary (Bellman optimality): if π is greedy for its value Q_π then it is optimal
  - In a finite MDP, the iteration $\pi \xrightarrow{\text{evaluate}} Q_\pi \xrightarrow{\text{greedy}} \pi$ converges, and then π is optimal

The RL scheme



Policy Iteration

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max

- If we know the MDP (model-based), we can just alternate evaluate/greedy:

Algorithm Policy Iteration

Initialize some policy π

repeat

Evaluate the policy $Q(s,a) \leftarrow \mathbb{E}_{\xi \sim p_\pi}[R | s_0 = s, a_0 = a]$

Update to the greedy policy $\pi(s) \leftarrow \arg\max_a Q(s,a)$

- Upon convergence, $\pi = \pi^*$ and $Q = Q^*$

Value Iteration

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- We can also alternate evaluate/greedy inside the loop over states:

Algorithm Value Iteration

Initialize some value function V

repeat

  for each state s

    Update $V(s) \leftarrow \max_a (r(s,a) + \gamma\, \mathbb{E}_{(s'|s,a)\sim p}[V(s')])$

- Must update each state repeatedly until convergence

- Upon convergence, $\pi^*(s) = \arg\max_a (r(s,a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[V(s')])$

Generalized Policy Iteration

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π'

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- We can even alternate in any order we wish:

$$V(s) \leftarrow \mathbb{E}_{(a|s)\sim\pi}[r(s,a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[V(s')]]$$

$$\pi(s) \leftarrow \arg\max_a (r(s,a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[V(s')])$$

- As long as each state gets each of the two update without starvation
  - The process will eventually converge to V^* and π^*

Model-free reinforcement learning

- We can be model-free using MC policy evaluation:

Algorithm MC model-free RL

Initialize some policy π

repeat

  Initialize some value function Q

  repeat to convergence

    Sample $\xi \sim p_\pi$

    Update $Q(s_t, a_t) \to R_{\geq t}(\xi)$ for all $t \geq 0$

  $\pi(s) \leftarrow \arg\max_a Q(s, a)$ for all s

- On-policy policy evaluation in the inner loop — very inefficient

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Off-policy model-free reinforcement learning

- Value iteration is model-based: $V(s) \leftarrow \max_a(r(s,a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[V(s')])$
- Action-value version: $Q(s,a) \leftarrow r(s,a) + \gamma \mathbb{E}_{(s'|s,a)\sim p}[\max_{a'} Q(s',a')]$
- A model-free (data-driven) version — Q-Learning:
  - On off-policy data (s,a,r,s'), update

$$Q(s,a) \to r + \gamma \max_{a'} Q(s',a')$$

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π'

max

[Watkins and Dayan, 1992]

Recap

- RL is a (policy evaluation ↔ policy improvement) loop
- Policy evaluation: model-based, Monte Carlo, or Temporal-Difference
  - Temporal-Difference exploits the sequential structure using dynamic programming
- TD can be off-policy by considering the action-value Q function
  - Off-policy data can be thrown out less often as the policy changes
- Policy improvement can be greedy
  - Arbitrarily alternated with policy evaluation of any kind (MB, MC, or TD)
- Many approaches can be made differentiable for Deep RL